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An N-Dimensional Analogue of Szegö's Limit Theorem

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Let $\mathbb{T}$ denote the unit circle in the complex plane, let $\phi \in L^\infty(\mathbb{T}^N)$, and for $(a_1, \ldots, a_N) \in (\mathbb{Z}_+ \setminus 0)^N$ let Λ be the rectangular lattice: $\Lambda = \{(m_1, \ldots, m_N) \in \mathbb{Z}^N$: $m_i \in [0, a_i)\}$. For each positive integer p define $\Lambda_p = \{(m_1, \ldots, m_N) \in \mathbb{Z}^N$: $m_i \in [0, pa_i)\}$. The Toeplitz matrix $T_p(\phi)$: $l^2(\Lambda_p) \to l^2(\Lambda_p)$ with symbol ϕ is defined by

$$T_p(\phi)\alpha(m) = \sum_{n \in \Lambda_p} \hat{\phi}(m-n)\alpha(n)$$

where $\{\hat{\phi}(m)\}_{m \in \mathbb{Z}^N}$ denotes the Fourier coefficients of ϕ. Assuming appropriate conditions on ϕ and on a function F we find $N+1$ terms of the asymptotic expansion of the trace of $F(T_p(\phi))$ as $p \to \infty$. This expansion takes the form

$$\operatorname{tr} F\big(T_p(\phi)\big) = \sum_{j=0}^{N} c_{J,F} P^{N-J} + o(1),$$

where the coefficients $c_{J,F} = c_{J,F}(\phi)$ depend on the $N-J$ dimensional faces of Λ. We also find an expansion for the case when the edges of Λ expand at different rates and we apply this generalization to compute an example.

1. INTRODUCTION

Let $\mathbb{T}$ denote the unit circle in the complex plane, let $\phi \in L^\infty(\mathbb{T}^N)$, and let Λ be the rectangular lattice

$$\Lambda = \big\{(m_1, \ldots, m_N) \in \mathbb{Z}_+^N : m_i \in \big[0, a_i\big)\big\}$$

where $\mathbb{Z}_+ = \{m \in \mathbb{Z}\colon\ m \geq 0\}$ and $(a_1, \ldots, a_N) \in (\mathbb{Z}_+ \setminus 0)^N$. For any $p = (p_1, \ldots, p_N) \in (\mathbb{Z}_+ \setminus 0)^N$ define

$$\Lambda_p = \big\{(m_1, \ldots, m_N) \in \mathbb{Z}_+^N : m_i \in \big[0, p_i a_i\big)\big\}.$$

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The Toeplitz matrix $T_p(\phi)$: $l^2(\Lambda_p) \to l^2(\Lambda_p)$ with symbol ϕ is defined by

$$T_p(\phi)\alpha(m) = \sum_{n \in \Lambda_p} \hat{\phi}(m-n)\alpha(n),$$

where $\{\hat{\phi}(m)\}_{m \in \mathbb{Z}^N}$ are the Fourier coefficients of ϕ

$$\hat{\phi}(m) = \left(\frac{1}{2\pi}\right)^N \int_0^{2\pi} \cdots \int_0^{2\pi} \phi(e^{it_1}, \dots, e^{it_N}) e^{-im \cdot t} dt_1 \cdots dt_N.$$

Note that in the future, for simplicity of notation, we shall sometimes write $\phi(t)$ instead of $\phi(e^{it_1}, \dots, e^{it_N})$.

Szegö considered the case when $N = 1$. An early version of his classic theorem [4] states that if ϕ is real and bounded and if F: $[\operatorname{ess\,inf} \phi, \operatorname{ess\,sup} \phi] \to \mathbb{C}$ is Riemann integrable, then the trace of $F(T_p(\phi))$ is given asymptotically as $p \to \infty$ by

$$\operatorname{tr} F\big(T_p(\phi)\big) = p\big(F(\phi)\big)\hat{}\,(0) + o(p). \tag{1.1}$$

Later, assuming that ϕ is positive and smooth, Szegö [5] found two terms of the asymptotics for the case when $F(z) = \log(z)$. Similar asymptotic expansions were then discovered for more general functions ϕ and F. In this paper we consider an N-dimensional analogue. Assuming appropriate conditions on our functions ϕ and F we find $N+1$ terms of the asymptotic expansion for the trace of $F(T_p(\phi))$ as $\inf p_i \to \infty$. In the special case when $p = (p, p, \dots, p)$ this expansion takes the form

$$\operatorname{tr} F\big(T_p(\phi)\big) = \sum_{J=0}^{N} c_J(F, \phi) p^{N-J} + o(1). \tag{1.2}$$

Seghier [3] also considered an N-dimensional analogue of Szegö's theorem. He let $\phi = 1/|P|^2$ where P is a polynomial and found $N+1$ terms of the expansion for trace of $(T_p(\phi))^{-1}$ as $\inf p_i \to \infty$. His formula is quite complicated. In contrast, by applying a generalization of formula (1.2) we replace $\phi = 1/|P|^2$ with the more general $\phi = gh$ where $g^{\pm 1}$ and $(\overline{h})^{\pm 1}$ are appropriately smooth functions contained in the Hardy space

$$H^\infty(\mathbb{T}^N) = \left\{ f \in L^\infty(\mathbb{T}) : \hat{f}(m) = 0 \text{ for } m \notin \mathbb{Z}_+^N \right\}$$

and we obtain the relatively simple result

$$\operatorname{tr}\big(T_p(\phi)\big)^{-1} = \sum_{m \in \mathbb{Z}_+^N} \left(\frac{1}{g}\right)\hat{}(m)\left(\frac{1}{h}\right)\hat{}(-m) \prod_{i=1}^{N} (a_i p_i - m_i) + o(1). \tag{1.3}$$

However, in order to use formula (1.3) we need to assume that

$$|\hat{\phi}(0)| > \sum_{m \in \mathbb{Z}^N \setminus 0} |\hat{\phi}(m)|.$$

2. NOTATION AND STATEMENT OF THE MAIN THEOREM

Let $\phi \in L^\infty(\mathbb{T}^N)$ and let F be analytic on the disk $D(0, \|\hat{\phi}\|_1 + \delta)$ for some $\delta > 0$. Assuming that ϕ satisfies condition $(*)$ (which will be defined shortly) we shall compute the first $N+1$ terms of the asymptotic expansion for the trace of $F(T_p(\phi))$ as $p \to \infty$. Our expansion takes the form

$$\operatorname{tr}\Big(F\big(T_p(\phi)\big)\Big) = \sum_{J=0}^{N} c_{J,F}\, p^{N-J} + o(1) \tag{2.1}$$

where the coefficients $c_{J,F} = c_{J,F}(\phi)$ are independent of p. Note that $F(T_p(\phi))$ is well defined for any $\phi \in L^\infty(\mathbb{T}^N)$ since the spectrum of $T_p(\phi)$ is contained in the disk of analyticity of F. Furthermore, $F(T_p(\phi))$ is of trace class since $T_p(\phi)$ is a finite sized $|\Lambda_p| \times |\Lambda_p|$ matrix (where $|\Lambda_p| = \operatorname{card}(\Lambda_p)$). The goal of this section is to define $(*)$ and to define $c_{J,F}$. We start by introducing some notation:

Partition the set $\{1, 2, \ldots, N\}$ into two ordered subsets $\{k_1, k_2, \ldots, k_r\}$ and $\{k'_1, k'_2, \ldots, k'_{N-r})$. Given $k = \{k_1, k_2, \ldots, k_r\}$ and given any function $\phi\colon \mathbb{T}^N \to \mathbb{C}$ define $\phi_k\colon \mathbb{T}^r \times \mathbb{T}^{N-r} \to \mathbb{C}$ by

$$\phi_k\big((e^{iu_{k_1}}, \ldots, e^{iu_{k_r}}), (e^{iu_{k'1}}, \ldots, e^{iu_{k'N-r}})\big) = \phi(e^{iu_1}, \ldots, e^{iu_N}).$$

Similarly, given any $f\colon \mathbb{Z}^N \to \mathbb{C}$ define $f_k\colon \mathbb{Z}^r \times \mathbb{Z}^{N-r} \to \mathbb{C}$ by

$$f_k\big((l_{k_1}, \ldots, l_{k_r}), (l_{k'_1}, \ldots, l_{k'_{N-r}})\big) = f(l_1, \ldots, l_N).$$

For each $s \in \mathbb{T}^r$ define the function $\phi_{k,s}\colon \mathbb{T}^{N-r} \to \mathbb{C}$ by

$$\phi_{k,s}(t) = \phi_k(s, t)$$

and for each $t \in \mathbb{T}^{N-r}$ define $\phi_k^t\colon \mathbb{T}^r \to \mathbb{C}$ by

$$\phi_k^t(s) = \phi_k(s, t).$$

Similarly, define $f_{k,m}\colon \mathbb{Z}^{N-r} \to \mathbb{C}$ and $f_k^n\colon \mathbb{Z}^r \to \mathbb{C}$ so that

$$f_{k,m}(n) = f_k(m, n) = f_k^n(m).$$

We say that $\phi\colon \mathbb{T}^N \to \mathbb{C}$ satisfies $(*)$ if for any $r \in \{0, 1, 2, \ldots, N\}$ and for any $k = \{k_1, k_2, \ldots, k_r\}$ the following properties are satisfied:

$$\text{(i)}\quad g_1(s) = \int_{\mathbb{T}^{N-r}} |\phi_{k,s}(t)|\, dt \in L^\infty(\mathbb{T}^r);$$

$$\text{(ii)}\quad g_2(s) = \sum_{n \in \mathbb{Z}^{N-r}} \left|\hat{\phi}_{k,s}(n)\right| \in L^\infty(\mathbb{T}^r);$$

$$\text{(iii)}\quad g_3(s) = \sum_{n \in \mathbb{Z}^{N-r}} \left|\hat{\phi}_{k,s}(n)\right| |n|^{N-r} \in L^2(\mathbb{T}^r).$$

Note that if $\phi \in C^\infty(\mathbb{T}^N)$ then ϕ satisfies $(*)$.

Let $\Lambda_J = \{\Lambda_J^1, \Lambda_J^2, \ldots, \Lambda_J^{\binom{N}{J}2^J}\}$ denote the $(N-J)$-dimensional faces of Λ where each face is made up of lattice points. Associate with each face, Λ_J^d, J-orthonormal vectors $\{\nu_{d_1}, \nu_{d_2}, \ldots, \nu_{d_J}\}$ in $\mathbb{C}^N$ defined such that:

(i) each $\nu_{d_J} \in \{\pm \mu_k\}_{k=1}^N$ where $\{\mu_k\}_{k=1}^N$ is the standard basis in $\mathbb{C}^N$;

(ii) each ν_{d_J} is normal to the face Λ_J^d and if the base point of ν_{d_j} is in Λ_J^d then its terminal point is in Λ.

To illustrate consider the case when $N = 3$. Then $\Lambda = \Lambda_0$ is the set of lattice points that make up a rectangular box (Fig. 1), $\Lambda_1 = \{\Lambda_1^1, \Lambda_1^2, \ldots, \Lambda_1^{-6}\}$ are the lattice points that make up the 6 faces of Λ (Fig. 2),

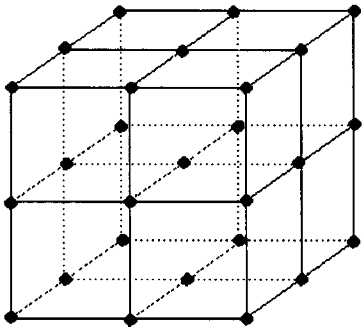

FIG. 1

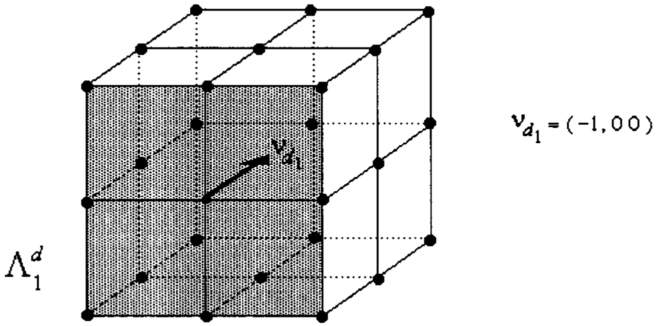

FIG. 2

$\Lambda_2 = \{\Lambda_2^1, \Lambda_2^2, \ldots, \Lambda_2^{12}\}$ are the lattice points that make up the 12 edges of Λ (Fig. 3), and $\Lambda_3 = \{\Lambda_3^1, \Lambda_3^2, \ldots, \Lambda_3^8\}$ are the 8 vertices of Λ (Fig. 4).

Next define $P_J^d = \prod_{j=1}^N A_j$ where

$$A_j = \begin{cases} \mathbf{T} & \text{if } \mu_j \in \{\nu_{d_1}, \ldots, \nu_{d_J}\}^{\perp} \\ \{0\} & \text{otherwise} \end{cases}.$$

So each P_J^d can be identified with $\mathbb{T}^{N-J}$. Given $t \in P_J^d$ define the function $\phi_{J,d,t}\colon \mathbb{T}^J \to \mathbb{C}$ by

$$\phi_{J,d,t}(e^{is_1}, \ldots, e^{is_J}) = \phi\left(t + \sum_{j=1}^{J} e^{i(\mu_{d_j}\cdot\nu_{d_j})s_j}\mu_{d_j}\right),$$

where μ_{d_j} is the basis vector corresponding to ν_{d_j}. In other words, $\mu_{d_j} = \pm\nu_{d_j}$. For example, suppose $N = 3$ and suppose Λ_2^d is the edge

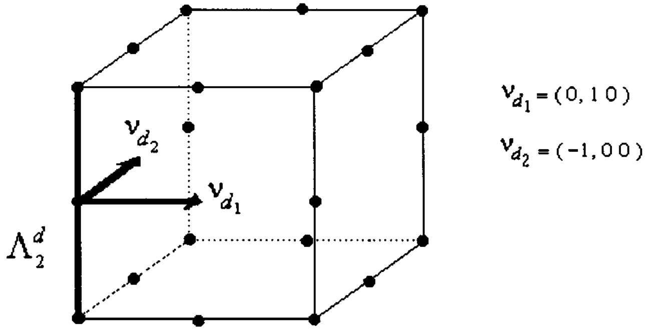

FIG. 3

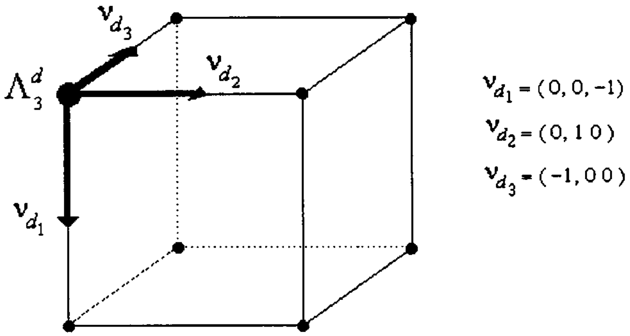

FIG. 4

shown in Fig. 3. Then $\mu_{d_1} = \nu_{d_1} = (0,1,0)$, $\mu_{d_2} = -\nu_{d_2} = (1,0,0)$, and $P_2^d = \{0\} \times \{0\} \times \mathbb{T}$. So for $t = (0,0,e^{it}) \in P_2^d$

$$\begin{aligned}\phi_{2,d,t}(e^{is_1}, e^{is_2}) &= \phi\Big(t + e^{i(\mu_{d_1}\cdot\nu_{d_1})s_1}\mu_{d_1} + e^{i(\mu_{d_2}\cdot\nu_{d_2})s_2}\mu_{d_2}\Big)\\ &= \phi\big((0,0,e^{it}) + (0,e^{is_1},0) + (e^{-is_2},0,0)\big)\\ &= \phi(e^{-is_2}, e^{is_1}, e^{it}).\end{aligned}$$

Each coefficient $c_{J,f}$ is of the form

$$c_{J,F} = (-1)^J\left(\frac{1}{2\pi}\right)^{N-J} \sum_{\Lambda_J^d \in \Lambda_J} |\Lambda_J^d| \int_{P_J^d} \operatorname{tr}\big(A_{\phi_{J,d,t},F}\big)\,dt$$

where $A_{\phi_{J,d,t},F}$ is a trace class operator on $l^2(\mathbb{Z}_+^J)$. In order to define this operator we need a few preliminary definitions.

Let $\theta \in L^\infty(\mathbb{T}^J)$ and let F be analytic on the disk $D(0, \|\theta\|_\infty + \delta)$ for some $\delta > 0$. The Toeplitz operator, $T(\theta)$: $l^2(\mathbb{Z}_+^J) \to l^2(\mathbb{Z}_+^J)$, with symbol θ is defined by

$$T(\theta)f = P\big(\theta\check{f}\big)^{\wedge},$$

where $\wedge$: $L^2(\mathbb{T}^J) \to l^2(\mathbb{Z}^J)$ is the Fourier transform, $\vee$ is the inverse Fourier transform, and P: $l^2(\mathbb{Z}^J) \to l^2(\mathbb{Z}_+^J)$ is the projection map. Note that if $f \in l^2(\mathbb{Z}^J)$ then $T(\theta)$ is defined by

$$T(\theta)f = P\big(\theta(Pf)^{\vee}\big)^{\wedge}.$$

Given $k = \{k_1, k_2, \ldots, k_r\}$ an ordered subset of $\{1, 2, \ldots, J\}$, define the operator $S_{k,\theta,F}$ on $l^2(\mathbb{Z}_+^J)$ by

$$S_{k,\theta,F}(f) = S_{k,\theta,F}(f_k) = P\Big(F\big(T(\theta_k^t)\big)\check{f}_k^2\Big)^{\wedge 2},$$

where $\wedge_2$ is the partial Fourier transform ($\vee_2$ is the partial inverse Fourier transform) with respect to the last $J-r$ variables (t being the new variable) and $T(\theta_k^t)$ is the Toeplitz operator acting only on the first r variables. In other words,

$$S_{k,\theta,F}\big(f_k(m,n)\big) = PG^{\wedge_2}(m,n) \qquad \text{where } G(m,t) = F\big(T\big(\theta_k^t\big)\big)\check{f}_k^2(m,t).$$

It is not hard to show that $S_{k,\theta,F}$ is a bounded operator on $l^2(\mathbb{Z}_+^J)$ with the properties that if $r=0$ then $S_{k,\theta,F} = T(F(\theta))$, if $r=J$ then $S_{k,\theta,F} = F(T(\theta))$, and if $F(z)=z$ then $S_{k,\theta,F} = T(\theta)$. Finally, we are ready to define the operator $A_{\theta,F}$ on $l^2(\mathbb{Z}_+^J)$ by

$$A_{\theta,F} = T(F(\theta)) + \sum_{r=1}^{J} (-1)^r \sum_{1\le k_1<\cdots<k_r\le J} S_{k,\theta,F}.$$

We now state the main theorem:

THEOREM 1. *Suppose* $\phi\colon \mathbb{T}^N \to \mathbb{C}$ *satisfies condition* $(*)$ *and suppose* F *is analytic on* $D(0,\|\hat{\phi}\|_1+\delta)$ *for some* $\delta>0$. *Then*

$$\operatorname{tr}\big(F\big(T_p(\phi)\big)\big) = \sum_{J=0}^{N} c_{J,F}\,p^{N-J} + o(1) \qquad \text{as } p\to\infty$$

where

$$c_{J,F} = (-1)^J\left(\frac{1}{2\pi}\right)^{N-J} \sum_{\Lambda_J^d\in\Lambda_J} |\Lambda_J^d| \int_{P_J^d} \operatorname{tr}\big(A_{\phi_{J,d,t},F}\big)\,dt.$$

Note that each $A_{\phi_{J,d,t},F}$ is a well defined operator on $l^2(\mathbb{Z}_+^J)$ since ϕ satisfying $(*)$ implies that $\phi_{J,d,t}\in L^\infty(\mathbb{T}^J)$ and since $D(0,\|\phi_{J,d,t}\|_\infty+\delta)\subset D(0,\|\hat{\phi}\|_1+\delta)$. The proof that $A_{\phi_{J,d,t},F}$ is of trace class will be shown in Lemma 3.6.

3. PROOF OF THEOREM 1

First we prove Theorem 1 for the case when F is a polynomial of the form $F(z)=z^M$. If $M=0$ then

$$c_{J,F} = \begin{cases} 0 & \text{if } J>0 \ \big(\text{since } A_{\theta,F}\colon \mathbb{Z}_+^J\to\mathbb{C} \text{ is the} \\ & \quad\ \text{zero operator for } J>0\big) \\ |\Lambda| & \text{if } J=0 \end{cases}$$

and if $M = 1$ then

$$c_{J,F} = \begin{cases} 0 & \text{if } J > 0 \ \big(\text{since } A_{\theta,F} \colon \mathbb{Z}_+^J \to \mathbb{C} \text{ is the} \\ & \quad \text{zero operator for } J > 0\big) \\ |\Lambda|\, \hat{\phi}(0) & \text{if } J = 0 \end{cases}$$

In both cases we get $\operatorname{tr}(F(T_p(\phi))) = \sum_{J=0}^N c_{J,F}\ p^{N-J}$. Therefore we need only to consider the case when $M > 1$. Our plan is to show that if $M > 1$ and if $\phi \colon \mathbb{T}^N \to \mathbb{C}$ satisfies $(*)$ then

$$\left| \operatorname{tr}\big(T_p(\phi)\big)^M - \sum_{J=0}^N c_{J,F}\, p^{N-J} \right| \\ \leq c(M-1)^{N+1} \|\hat{\phi}\|_1^M \sum_{|m| \geq pa/(M-1)} |\hat{\phi}(m)|\, |m|^N,$$

where a and c are constants independent of p and M. The proof of the above statement is accomplished by a series of nine lemmas.

We start by finding a formula for the trace of the operator $(T_p(\phi))^M$.

LEMMA 3.1. *Assume $\phi \in L^1(\mathbb{T}^N)$ and $\hat{\phi} \in l^1(\mathbb{Z}^N)$. Then*

$$\operatorname{tr}\big(T_p(\phi)\big)^M = \sum_{m_1,\dots,m_{M-1} \in \mathbb{Z}^N} \hat{\phi}(m_1) \cdots \hat{\phi}(m_{M-1})\, \hat{\phi}(-m_1 - \cdots - m_{M-1}) \\ \times |\Lambda_p \cap \big(\Lambda_p + m_1\big) \cap \cdots \\ \cap \big(\Lambda_p + m_1 + \cdots + m_{M-1}\big)|.$$

Proof. The kernel of $(T_p(\phi))^M$ at a point $(m, n) \in \Lambda_p \times \Lambda_p$ is equal to

$$\sum_{m_1,\dots,m_{M-1} \in \mathbb{Z}^N} \hat{\phi}(m - m_1)\, \hat{\phi}(m_1 - m_2) \cdots \hat{\phi}(m_{M-2} - m_{M-1}) \\ \times \hat{\phi}(m_{M-1} - n)\, \chi_{\Lambda_p}(m_1) \cdots \chi_{\Lambda_p}(m_{M-1}),$$

where χ_{Λ_p} is the characteristic function of the set Λ_p. Change variables, replacing $(m - m_1)$, $(m_1 - m_2), \dots, (m_{M-2} - m_{M-1})$ by $m_1, \dots, m_{M-1}$ respectively. We obtain

$$\sum_{m_1,\dots,m_{M-1} \in \mathbb{Z}^N} \hat{\phi}(m_1)\, \hat{\phi}(m_2) \cdots \hat{\phi}(m_{M-1}) \\ \times \hat{\phi}(m - n - m_1 - m_2 - \cdots - m_{M-1}) \\ \times \chi_{\Lambda_p}(m - m_1) \cdots \chi_{\Lambda_p}(m - m_1 - \cdots - m_{M-1}).$$

To find the trace of the operator we sum the kernel over the diagonal of $\Lambda_p \times \Lambda_p$. The result is

$$\begin{aligned}\operatorname{tr}\big(T_p(\phi)\big)^M = \sum_{m\in\Lambda_p} \sum_{m_1,\dots,m_{M-1}\in\mathbb{Z}^N} & \hat{\phi}(m_1)\cdots\hat{\phi}(m_{M-1}) \\ & \times \hat{\phi}(-m_1-\cdots-m_{M-1}) \\ & \times \chi_{\Lambda_p}(m-m_1)\cdots\chi_{\Lambda_p}(m-m_1-\cdots-m_{M-1}).\end{aligned}$$

By Fubini's theorem we can switch the order of summation to obtain

$$\begin{aligned}&\sum_{m_1,\dots,m_{M-1}\in\mathbb{Z}^N} \hat{\phi}(m_1)\cdots\hat{\phi}(m_{M-1})\hat{\phi}(-m_1-\cdots-m_{M-1}) \\ &\quad\times \sum_{m\in\Lambda_p} \chi_{\Lambda_p}(m-m_1)\cdots\chi_{\Lambda_p}(m-m_1-\cdots-m_{M-1}) \\ &\quad= \sum_{m_1,\dots,m_{M-1}\in\mathbb{Z}^N} \hat{\phi}(m_1)\cdots\hat{\phi}(m_{M-1})\hat{\phi}(-m_1-\cdots-m_{M-1}) \\ &\qquad\times |\Lambda_p \cap (\Lambda_p+m_1)\cap\cdots\cap(\Lambda_p+m_1+\cdots+m_{M-1})|. \qquad \blacksquare\end{aligned}$$

Next we want to study the kernel of the operators $A_{\phi_{J,d,t},F}$. The following two lemmas will help.

LEMMA 3.2. *Assume $\theta \in L^1(\mathbb{T}^J)$ and $\hat{\theta}\in l^1(\mathbb{Z}^J)$. Then*

(a) *the operator $(T(\theta))^M$ has kernel*

$$\begin{aligned}\sum_{m_1,\dots,m_{M-1}\in\mathbb{Z}^J} & \hat{\theta}(m_1)\cdots\hat{\theta}(m_{M-1}) \\ & \times \hat{\theta}(m-n-m_1-\cdots-m_{M-1}) \\ & \times \chi_{\mathbb{Z}^J_+}(m-m_1)\cdots\chi_{\mathbb{Z}^J_+}(m-m_1-\cdots-m_{M-1})\end{aligned}$$

where $\chi_{\mathbb{Z}^J_+}$ is the characteristic function of the set $\mathbb{Z}^J_+$.

(b) *the operator $T(\theta^M)$ has kernel*

$$\begin{aligned}\sum_{m_1,\dots,m_{M-1}\in\mathbb{Z}^J} & \hat{\theta}(m_1)\cdots\hat{\theta}(m_{M-1}) \\ & \times \hat{\theta}(m-n-m_1-\cdots-m_{M-1}).\end{aligned}$$

Proof of Part (a). The assumptions on θ imply that each of the terms in the following string of equalities are well defined:

$$T(\theta)f(m) = P\big(\theta\check{f}\big)\hat{}\,(m) = P\big(\hat{\theta}*f\big)(m) = \sum_{n\in\mathbb{Z}^J_+}\hat{\theta}(m-n)f(n).$$

Therefore

$$\begin{aligned}
(T(\theta))^{M} f(m) &= \sum_{m_1 \in \mathbb{Z}_+^J} \hat{\theta}(m - m_1)(T(\theta))^{M-1} f(m_1) \\
&= \sum_{m_1, m_2 \in \mathbb{Z}_+^J} \hat{\theta}(m - m_1)\hat{\theta}(m_1 - m_2)(T(\theta))^{M-2} f(m_2) \\
&\quad \dots \\
&= \sum_{m_1, \dots, m_{M-1} \in \mathbb{Z}_+^J} \hat{\theta}(m - m_1)\hat{\theta}(m_1 - m_2) \cdots \\
&\quad \times \hat{\theta}(m_{M-1} - n) f(n) \\
&= \sum_{m_1, \dots, m_{M-1} \in \mathbb{Z}^J} \hat{\theta}(m - m_1)\hat{\theta}(m_1 - m_2) \cdots \\
&\quad \times \hat{\theta}(m_{M-1} - n)\chi_{\mathbb{Z}_+^J}(m_1) \cdots \chi_{\mathbb{Z}_+^J}(m_{M-1}) f(n).
\end{aligned}$$

The conclusion follows by a change of variables: replace $(m - m_1)$, $(m_1 - m_2), \dots, (m_{M-2} - m_{M-1})$ by $m_1, \dots, m_{M-1}$ respectively.

Proof of Part (b).

$$\begin{aligned}
T(\theta^M) &= \sum_{n \in \mathbb{Z}_+^J} (\theta^M)\hat{}\,(m - n) f(n) \\
&= \sum_{n \in \mathbb{Z}_+^J} \hat{\theta} * (\theta^{M-1})\hat{}\,(m - n) f(n) \\
&= \sum_{n \in \mathbb{Z}_+^J} \sum_{m_1 \in \mathbb{Z}^J} \hat{\theta}(m - n - m_1)(\theta^{M-1})\hat{}\,(m_1) f(n) \\
&= \sum_{n \in \mathbb{Z}_+^J} \sum_{m_1 \in \mathbb{Z}^J} \hat{\theta}(m - n - m_1)\hat{\theta} * (\theta^{M-2})\hat{}\,(m_1) f(n) \\
&= \sum_{n \in \mathbb{Z}_+^J} \sum_{m_1, m_2 \in \mathbb{Z}^J} \hat{\theta}(m - n - m_1) \\
&\quad \times \hat{\theta}(m_1 - m_2) * (\theta^{M-2})\hat{}\,(m_2) f(n) \\
&\quad \dots \\
&= \sum_{n \in \mathbb{Z}_+^J} \sum_{m_1, \dots, m_{M-1} \in \mathbb{Z}^J} \\
&\quad \times \hat{\theta}(m - n - m_1)\hat{\theta}(m_1 - m_2) \cdots \hat{\theta}(m_{M-2} - m_{M-1}) f(n).
\end{aligned}$$

The conclusion follows by a change of variables: replace $(m - n - m_1)$, $(m_1 - m_2), \dots, (m_{M-2} - m_{M-1})$ by $m_1, \dots, m_{M-1}$ respectively. ∎

LEMMA 3.3. *Let* $\theta\colon \mathbb{T}^J \to \mathbb{C}$ *satisfy condition* $(*)$, *let* $F(z) = z^M$, *and let* $k = (k_1, k_2, \ldots, k_r)$ *then* $S_{k,\theta,F}\colon l^2(\mathbb{Z}_+^J) \to l^2(\mathbb{Z}_+^J)$ *has kernel*

$$\sum_{m_1,\ldots,m_{M-1}\in\mathbb{Z}^J} \hat{\theta}(m_1)\cdots\hat{\theta}(m_{M-1}) \times \hat{\theta}(m-n-m_1-\cdots-m_{M-1}) \times \prod_{i=1}^{r} \chi_{k_i}(m-m_1)\cdots\chi_{k_i}(m-m_1-\cdots-m_{M-1})$$

where χ_i *is the characteristic function of the set*

$$\big\{n = (n_1,\ldots,n_J) \in \mathbb{Z}^J\colon n_i \geq 0\big\}.$$

Proof. Let $f \in l^2(\mathbb{Z}_+^J)$ and let $m \in \mathbb{Z}_+^J$. Choose $(j,l) \in \mathbb{Z}_+^r \times \mathbb{Z}_+^{J-r}$ such that

$$(j,l) = \big((m_{k_1},\ldots,m_{k_r}),(m_{k'_1},\ldots,m_{k'_{J-r}})\big).$$

Then

$$\begin{aligned}
S_{k,\theta,F}f(m) &= S_{k,\theta,F}f_k(j,l) = PG^{\wedge_2}(j,l) \\
&\qquad\qquad \Big(\text{where } G(j,t) = \big(T\big(\theta_k^t\big)\big)^M \check{f}_k^2(j,t)\Big) \\
&= \left(\frac{1}{2\pi}\right)^{J-r} \int_{\mathbb{T}^{J-r}} e^{-il\cdot t}\big(T\big(\theta_k^t\big)\big)^M \check{f}_k^2(j,t)\,dt \\
&= \left(\frac{1}{2\pi}\right)^{J-r} \int_{\mathbb{T}^{J-r}} e^{-il\cdot t}\big(T\big(\theta_k^t\big)\big)^M \Bigg(\sum_{\tilde{l}\in\mathbb{Z}_+^{J-r}} e^{i\tilde{l}\cdot t} f_k\big(j,\tilde{l}\big)\Bigg)\,dt \\
&= \left(\frac{1}{2\pi}\right)^{J-r} \int_{\mathbb{T}^{J-r}} \sum_{\tilde{l}\in\mathbb{Z}_+^{J-r}} e^{-it\cdot(l-\tilde{l})}\big(T\big(\theta_k^t\big)\big)^M f_k\big(j,\tilde{l}\big)\,dt. \qquad (3.1)
\end{aligned}$$

By applying Lemma 3.2a and by keeping in mind that $T(\theta_k^t)$ only acts on the first r variables we have that

$$\begin{aligned}
&\big(T\big(\theta_k^t\big)\big)^M f_k\big(j,\tilde{l}\big) \\
&= \sum_{\tilde{j}\in\mathbb{Z}_+^r} \sum_{j_1,\ldots,j_{M-1}\in\mathbb{Z}^r} \theta_k^{t\wedge}(j_1)\cdots\theta_k^{t\wedge}(j_{M-1})\,\theta_k^{t\wedge}\big(j-\tilde{j}-j_1-\cdots-j_{M-1}\big) \\
&\qquad \times \chi_{\mathbb{Z}_+^r}(j-j_1)\cdots\chi_{\mathbb{Z}_+^r}(j-j_1-\cdots-j_{M-1}) f_k\big(\tilde{j},\tilde{l}\big).
\end{aligned}$$

Make the following substitutions:

$$\begin{aligned} g_1(t) &= \left(\theta_k^t\right)^{\wedge}(j_1) \\ g_2(t) &= \left(\theta_k^t\right)^{\wedge}(j_2) \\ &\cdots \\ g_{M-1}(t) &= \left(\theta_k^t\right)^{\wedge}(j_{M-1}) \\ g_M(t) &= \left(\theta_k^t\right)^{\wedge}\left(j - \tilde{j} - j_1 - \cdots - j_{M-1}\right). \end{aligned}$$

Since θ satisfies $(*)$, Fubini's theorem applies and expression (3.1) is equal to

$$\sum_{\tilde{j}\in \mathbb{Z}_+^r} \sum_{\tilde{l}\in \mathbb{Z}_+^{J-r}} \sum_{j_1,\ldots,j_{M-1}\in \mathbb{Z}^r} \left(\left(\frac{1}{2\pi}\right)^{J-r} \int_{\mathbb{T}^{J-r}} e^{-it\cdot(l-\tilde{l})} g_1 g_2 \cdots g_M(t)\, dt\right)$$
$$\times \chi_{\mathbb{Z}_+^r}(j - j_1) \cdots \chi_{\mathbb{Z}_+^r}(j - j_1 - \cdots - j_{M-1}) f_k\left(\tilde{j}, \tilde{l}\right). \quad (3.2)$$

The expression inside the above parenthesis is equal to

$$\begin{aligned} &(g_1 g_2 \cdots g_M)^{\wedge}(l - \tilde{l}) \\ &\quad = \hat{g}_1 * \hat{g}_2 * \cdots * \hat{g}_M(l - \tilde{l}) \\ &\quad = \sum_{l_1,\ldots,l_{M-1}\in \mathbb{Z}^{J-r}} \hat{g}_1(l_1)\hat{g}_2(l_2) \cdots \hat{g}_{M-1}(l_{M-1}) \\ &\qquad \times \hat{g}_M\left(l - \tilde{l} - l_1 - \cdots - l_{M-1}\right) \\ &\quad = \sum_{l_1,\ldots,l_{M-1}\in \mathbb{Z}^{J-r}} \hat{\theta}_k(j_1, l_1) \cdots \hat{\theta}_k(j_{M-1}, l_{M-1}) \\ &\qquad \times \hat{\theta}_k\left(j - \tilde{j} - j_1 - \cdots - j_{M-1}, l - \tilde{l} - l_1 - \cdots - l_{M-1}\right). \end{aligned}$$

Therefore (3.2) is equal to

$$\begin{aligned} &\sum_{\tilde{m}\in \mathbb{Z}_+^J} \sum_{m_1,\ldots,m_{M-1}\in \mathbb{Z}^J} \hat{\theta}(m_1) \cdots \hat{\theta}(m_{M-1}) \\ &\quad \times \hat{\theta}(m - \tilde{m} - m_1 - \cdots - m_{M-1}) \\ &\quad \times \prod_{i=1}^{r} \chi_{k_i}(m - m_1) \cdots \chi_{k_i}(m - m_1 - \cdots - m_{M-1}) f(\tilde{m}). \end{aligned}$$

∎

LEMMA 3.4. *Let* $F(z) = z^M$ *and let* $\theta\colon \mathbb{T}^J \to \mathbb{C}$ *satisfy condition* $(*)$. *Then* $A_{\theta, F}\colon l^2(\mathbb{Z}_+^J) \to l^2(\mathbb{Z}_+^J)$ *has kernel*

$$\sum_{m_1, \ldots, m_{M-1} \in \mathbb{Z}^J} \hat{\theta}(m_1) \cdots \hat{\theta}(m_{M-1}) \times \hat{\theta}(m - n - m_1 - \cdots - m_{M-1}) \times \prod_{i=1}^{J} \left(1 - \chi_i(m - m_1) \cdots \chi_i(m - m_1 - \cdots - m_{M-1})\right).$$

Proof. Let $a_i = \chi_i(m - m_1) \cdots \chi_i(m - m_1 - \cdots - m_{M-1})$. Then by Lemmas 3.2b and 3.3 and by the fact that

$$A_{\theta, F} = T(F(\theta)) + \sum_{r=1}^{J} (-1)^r \sum_{1 \le k_1 < \cdots < k_r \le J} S_{k, \theta, F}$$

we know that $A_{\theta, F} f(m) = \sum_{n \in \mathbb{Z}_+^J} K(m, n) f(n)$ where

$$K(m, n) = \sum_{m_1, \ldots, m_{M-1} \in \mathbb{Z}^J} \hat{\theta}(m_1) \cdots \hat{\theta}(m_{M-1}) \times \hat{\theta}(m - n - m_1 - \cdots - m_{M-1}) \times \left(1 + \sum_{r=1}^{J} (-1)^r \sum_{1 \le k_1 < \cdots < k_r \le J} \prod_{i=1}^{r} a_{k_i}\right).$$

The result follows by applying the identity:

$$\prod_{i=1}^{J} (1 - a_i) = 1 + \sum_{r=1}^{J} (-1)^r \sum_{1 \le k_1 < \cdots < k_r \le J} \prod_{i=1}^{r} a_{ki}. \qquad \blacksquare$$

The next step is to show that, provided θ satisfies certain conditions, the operator $A_{\theta, F}$ is of trace class. We achieve this goal by first writing $A_{\theta, F}$ as a sum of operators which are easier to work with.

Define χ_i^* to be the characteristic function of the set $\{n = (n_1, n_2, \ldots, n_J) \in \mathbb{Z}^J\colon n_i > 0\}$ and recall that χ_i is the characteristic function of the set $\{n = (n_1, n_2, \ldots, n_J) \in \mathbb{Z}^J\colon n_i \ge 0\}$.

LEMMA 3.5. *$A_{\theta, F}$ is equal to the sum of operators each having a kernel of the form*

$$\sum_{m_1, \ldots, m_{M-1} \in \mathbb{Z}^J} \theta^{\wedge}(m_1) \cdots \theta^{\wedge}(m_{M-1})$$
$$\times \theta^{\wedge}(m-n-m_1-\cdots-m_{M-1}) \chi_{E_m}(m_1, \ldots, m_{M-1}) a_1^{p_1} a_2^{p_2} \cdots a_J^{p_J}$$

where $p_i \in \{1, 2, \ldots, M-1\}$, $a_i^p = \chi_i^*(m-m_1) \cdots \chi_i^*(m-m_1-\cdots-m_{p-1}) \chi_i^*(m_1+\cdots+m_p-m)$, and χ_{E_m} is the characteristic function of the set

$$E_m = \Big\{ (m_1, \ldots, m_{M-1}) \in (\mathbb{Z}^J)^{M-1}:$$
$$\prod_{i=1}^{J} (1-\chi_i(m-m_1) \cdots \chi_i(m-m_1-\cdots-m_{M-1}))$$
$$= \prod_{i=1}^{J} (1-\chi_i^*(m-m_1) \cdots \chi_i^*(m-m_1-\cdots-m_{M-1})) \Big\}.$$

Proof. By Lemma 3.4 the kernel of $A_{\theta, F}$ is equal to

$$\sum_{m_1, \ldots, m_{M-1} \in \mathbb{Z}^J} \hat{\theta}(m_1) \cdots \hat{\theta}(m_{M-1})$$
$$\times \hat{\theta}(m-n-m_1-\cdots-m_{M-1})$$
$$\times \prod_{i=1}^{J} (1-\chi_i(m-m_1) \cdots \chi_i(m-m_1-\cdots-m_{M-1})).$$
$$= \sum_{m_1, \ldots, m_{M-1} \in \mathbb{Z}^J} \hat{\theta}(m_1) \cdots \hat{\theta}(m_{M-1})$$
$$\times \hat{\theta}(m-n-m_1-\cdots-m_{M-1}) \chi_{E_m}(m_1, \ldots, m_{M-1})$$
$$\times \prod_{i=1}^{J} (1-\chi_i^*(m-m_1) \cdots \chi_i^*(m-m_1-\cdots-m_{M-1})).$$

Therefore it suffices to prove

$$\prod_{i=1}^{J} (1-\chi_i^*(m-m_1) \cdots \chi_i^*(m-m_1-\cdots-m_{M-1}))$$
$$= \sum_{1 \le p_1, \ldots, p_J \le M-1} a_1^{p_1} a_2^{p_2} \cdots a_J^{p_J}. \tag{3.3}$$

Clearly the LHS of Eq. (3.3) is either 0 or 1. We claim that the RHS is also either 0 or 1. To prove this claim note the following:

$a_i^p = 0$ or 1;

$a_i^p = 1$ and $p > q \Rightarrow \chi_i^*(m - m_1 - \cdots - m_q) = 1 \Rightarrow \chi_i^*(m_1 + \cdots + m_q - m) = 0 \Rightarrow a_i^q = 0$;

$a_i^p = 1$ and $p < q \Rightarrow \chi_i^*(m_1 + \cdots + m_p - m) = 1 \Rightarrow \chi_i^*(m - m_1 - \cdots - m_p) = 0 \Rightarrow a_i^q = 0$. Therefore $\{a_i^p = 1 \Rightarrow a_i^q = 0$ for $p \neq q\} \Rightarrow \{a_1^{p_1} a_2^{p_2} \cdots a_J^{p_J} = 1 \Rightarrow a_1^{q_1} a_2^{q_2} \cdots a_J^{q_J} = 0$ for $(p_1, p_2, \ldots, p_J) \neq (q_1, q_2, \ldots, q_J)\} \Rightarrow$ RHS $= 0$ or 1. So to prove the lemma it suffices to show LHS $= 0 \Leftrightarrow$ RHS $= 0$.

$(\Rightarrow)$ LHS $= 0 \Rightarrow \exists i$ such that $\chi_i^*(m - m_1 - \cdots - m_p) = 1$ for all $p \Rightarrow \chi_i^*(m_1 + m_2 + \cdots + m_p - m) = 0$ for all $p \Rightarrow a_i^p = 0$ for all $p \Rightarrow a_1^{p_1} a_2^{p_2} \cdots a_i^{p_i} \cdots a_J^{p_J} = 0 \Rightarrow$ RHS $= 0$.

$(\Leftarrow)$ Suppose LHS $\neq 0$. Then for each $i \in \{1, 2, \ldots, J\}$ we have $\chi_i^*(m - m_1) \cdots \chi_i^*(m - m_1 - \cdots - m_{M-1}) = 0$. So for each i choose p_i such that $\chi_i^*(m - m_1) \cdots \chi_i^*(m - m_1 - \cdots - m_{p_i - 1}) = 1$ and $\chi_i^*(m - m_1 - \cdots - m_{p_i}) = 0$. Then $a_i^{p_i} = \chi_i^*(m - m_1) \cdots \chi_i^*(m - m_1 - \cdots - m_{p_i - 1}) \chi_i^*(m_1 + \cdots + m_{p_i} - m) = 1$. Therefore by our choice of $\{p_1, p_2, \ldots, p_J\}$ we have $a_1^{p_1} a_2^{p_2} \cdots a_J^{p_J} = 1 \Rightarrow \sum_{1 \leq p_1, \ldots, p_J \leq M-1} a_1^{p_1} a_2^{p_2} \cdots a^{p_J} = 1 \Rightarrow$ RHS $\neq 0$. ∎

By working with these new operators we can prove the following lemma:

LEMMA 3.6. *Let $F(z) = z^M$ and let $\theta \colon \mathbb{T}^J \to \mathbb{C}$ satisfy condition* $(*)$. *Then $A_{\theta, F}$ is a trace class operator on $l^2(\mathbb{Z}_+^J)$ and*

$$\|A_{\theta, F}\|_1 \leq c(M-1)^{2J+1} \|\hat{\theta}\|_1^{M-3}$$

where $c = c(\theta)$ is independent of M.

Proof. By Lemma 3.5 the kernel of $A_{\theta, F}$ is equal to

$$\sum_{1 \leq p_1, \ldots, p_J \leq M-1} \sum_{m_1, \ldots, m_{M-1} \in \mathbb{Z}^J} \hat{\theta}(m_1) \cdots \hat{\theta}(m_{M-1})$$
$$\times \hat{\theta}(m - n - m_1 - \cdots - m_{M-1})$$
$$\times \chi_{E_m}(m_1, \ldots, m_{M-1}) a_1^{p_1} a_2^{p_2} \cdots a_J^{p_J}.$$

Therefore to prove that $A_{\theta, F}$ is of trace class it suffices to show that each term is trace class. In other words, for any $p = (p_1, p_2, \ldots, p_J)$ it suffices to show that the operator, A_p, with kernel

$$\sum_{m_1,\ldots,m_{M-1}\in\mathbb{Z}^J} \hat{\theta}(m_1)\cdots\hat{\theta}(m_{M-1}) \times \hat{\theta}(m-n-m_1-\cdots-m_{M-1}) \times \chi_{E_m}(m_1,\ldots,m_{M-1})a_1^{p_1}a_2^{p_2}\cdots a_J^{p_J}$$

is of trace class. Fix $m_2,\ldots,m_{M-1}$ and let $A_p(m_2,\ldots,m_{M-1})$ be the operator with kernel

$$K_{m_2,\ldots,m_{M-1}}(m,n) = \sum_{m_1\in\mathbb{Z}^J} \hat{\theta}(m_1)\cdots\hat{\theta}(m_{M-1}) \times\hat{\theta}(m-n-m_1-\cdots-m_{M-1}) \times \chi_{E_m}(m_1,\ldots,m_{M-1})a_1^{p_1}a_2^{p_2}\cdots a_J^{p_J}.$$

Then

$$A_p = \sum_{m_2,\ldots,m_{M-1}\in\mathbb{Z}^J} A_p(m_2,\ldots,m_{M-1}).$$

If we show

1. $A_p(m_2,\ldots,m_{M-1})$ is trace class and
2. $\sum_{m_2,\ldots,m_{M-1}\in\mathbb{Z}^J}\|A_p(m_2,\ldots,m_{M-1})\|_1 \le c(M-1)^{J+1}\|\hat{\theta}\|_1^{M-3}$

then it would follow that A_p is a trace class operator with

$$\|A_p\|_1 \le \sum_{m_2,\ldots,m_{M-1}\in\mathbb{Z}^J}\|A_p(m_2,\ldots,m_{M-1})\|_1 \le c(M-1)^{J+1}\|\hat{\theta}\|_1^{M-3}.$$

But this would also mean that

$$\|A_{\theta,F}\|_1 = \sum_{1\le p_1,\ldots,p_J\le M-1}\|A_p\|_1 \le c(M-1)^{2J+1}\|\hat{\theta}\|_1^{M-3}$$

which is the desired result. So it suffices to prove (1) and (2).

Proof of (1). The plan is to show that

$$K_{m_2,\ldots,m_{M-1}}(m,n) = \sum_{k\in\mathbb{Z}_+^J} P_{m_2,\ldots,m_{M-1}}(m,k)Q_{m_2,\ldots,m_{M-1}}(k,n)$$

where

$$\sum_{m,k\in\mathbb{Z}_+^J}|P_{m_2,\ldots,m_{M-1}}(m,k)|^2<\infty \text{ and } \sum_{m,k\in\mathbb{Z}_+^J}|Q_{m_2,\ldots,m_{M-1}}(m,k)|^2<\infty.$$

From this we can conclude that $A_p(m_2,\dots,m_{M-1})$ is equal to the product of two Hilbert−Schmidt operators and thus is trace class. We know

$$\begin{aligned} K_{m_2,\dots,m_{M-1}}(m,n) = \sum_{m_1\in\mathbb{Z}^J} & \hat\theta(m_1)\cdots\hat\theta(m_{M-1}) \\ & \times \hat\theta(m-n-m_1-\cdots-m_{M-1}) \\ & \times \chi_{E_m}(m_1,\dots,m_{M-1}) a_1^{p_1}a_2^{p_2}\cdots a_J^{p_J}. \end{aligned}$$

Make the following changes of variables: first in order to avoid a notational confusion replace m by m_0, then define $k=(k_1,k_2,\dots,k_J)$ such that

$$k_i = m_{1,i}+m_{2,i}+\cdots+m_{p_i,i}-m_{0,i}.$$

By replacing $m_{1,i}$ with $k_i+m_{0,i}-m_{2,i}-\cdots-m_{p_i,i}$ we get

$$K_{m_2,\dots,m_{M-1}}(m_0,n) = \sum_{k\in\mathbb{Z}_+^J} P_{m_2,\dots,m_{M-1}}(m_0,k)Q_{m_2,\dots,m_{M-1}}(k,n) \quad (3.4)$$

where

$$\begin{aligned} P_{m_2,\dots,m_{M-1}}(m_0,k) = & \hat\theta\big(\cdots,k_i+m_{0,i}-m_{2,i}-\cdots-m_{p_i,i},\dots,\big) \\ & \times \hat\theta(m_2)\cdots\hat\theta(m_{M-1})\chi_{E_{M_0}}\sqrt{a_1^{p_1}a_2^{p_2}\cdots a_J^{p_J}} \end{aligned}$$

and

$$\begin{aligned} & Q_{m_2,\dots,m_{M-1}}(k,n) \\ & \quad = \hat\theta\big(\cdots,-n_i-k_i-m_{p_i+1,i}-\cdots-m_{M-1,i},\cdots\big)\sqrt{a_1^{p_1}a_2^{p_2}\cdots a_J^{p_J}}. \end{aligned}$$

Note that in Eq. (3.4) we are summing over $k\in\mathbb{Z}_+^J$ instead of $\mathbb{Z}^J$. This is justified by the fact that $a_i^{p_i}=0$ whenever $\chi_i^*(m_1+\cdots+m_{p_i}-m)=\chi_i^*(k)=0$.

Claim.

(i)

$$\begin{aligned} & \sum_{m,k\in\mathbb{Z}_+^J} |P_{m_2,\dots,m_{M-1}}(m,k)|^2 \\ & \qquad \le c_1(M-1)^J\left(1+\sum_{i=2}^{M-1}|m_i|^J\right)|\hat\theta(m_2)\cdots\hat\theta(m_{M-1})|^2 \end{aligned}$$

(ii)

$$\sum_{m,k\in\mathbb{Z}_+^J} |Q_{m_2,\dots,m_{M-1}}(m,k)|^2 \le c_1(M-1)^J\left(1+\sum_{i=2}^{M-1}|m_i|^J\right)$$

where

$$c_1 = c_1(\theta) = \sum_{m\in\mathbb{Z}^J} |\hat{\theta}(m)| + \sum_{m\in\mathbb{Z}^J} |\hat{\theta}(m)|^2 + \sum_{m\in\mathbb{Z}^J} |\hat{\theta}(m)|\,|m|^J + \sum_{m\in\mathbb{Z}^J} |\hat{\theta}(m)|^2\,|m|^J.$$

Proof of Claim. Note that since the proof of parts (i) and (ii) are very similar we shall only prove part (i).

$$\sum_{m_0,k\in\mathbb{Z}_+^J} |P_{m_2,\dots,m_{M-1}}(m_0,k)|^2 = \sum_{\substack{m_0\in\mathbb{Z}_+^J\\ k\in(\mathbb{Z}_+\setminus 0)^J}} |P_{m_2,\dots,m_{M-1}}(m_0,k)|^2$$

(since $a_i^{p_i} = 0$ whenever $k_i \le 0$)

$$\le \sum_{\substack{m_0\in\mathbb{Z}_+^J\\ k\in(\mathbb{Z}_+\setminus 0)^J}} |\hat{\theta}(\dots,k_i + m_{0,i} - m_{2,i} - \cdots - m_{p_i,i},\dots)\hat{\theta}(m_2)\cdots\hat{\theta}(m_{M-1})|^2$$

$$= |\hat{\theta}(m_2)\cdots\hat{\theta}(m_{M-1})|^2 \times \sum_{\substack{m_0\in\mathbb{Z}_+^J\\ k\in(\mathbb{Z}_+\setminus 0)^J}} |\hat{\theta}(\dots,k_i + m_{0,i} - m_{2,i} - \cdots - m_{p_i,i},\dots)|^2.$$

Change variables, let $l_i = k_i + m_{0,i}$, and we obtain

$$= |\hat{\theta}(m_2)\cdots\hat{\theta}(m_{M-1})|^2 \times \sum_{m_0\in\mathbb{Z}_+^J} \left(\sum_{l_1=m_{0,1}+1}^{\infty} \cdots \sum_{l_J=m_{0,J}+1}^{\infty} |\hat{\theta}(\dots,l_i - m_{2,i} - \cdots - m_{p_i,i},\dots)|^2 \right)$$

$$= |\hat{\theta}(m_2)\cdots\hat{\theta}(m_{M-1})|^2 \times \sum_{l\in(\mathbb{Z}_+\setminus 0)^J} \left(\sum_{m_{0,1}=1}^{l_1} \cdots \sum_{m_{0,J}=1}^{l_J} |\hat{\theta}(\dots,l_i - m_{2,i} - \cdots - m_{p_i,i},\dots)|^2 \right)$$

$$= |\hat{\theta}(m_2)\cdots\hat{\theta}(m_{M-1})|^2 \times \sum_{l\in(\mathbb{Z}_+\setminus 0)^J} |l_1 l_2\cdots l_J|\,|\hat{\theta}(\dots,l_i - m_{2,i} - \cdots - m_{p_i,i},\dots)|^2.$$

Change variables again, let $m_{0,i} = l_i - m_{2,i} - \cdots - m_{p_i,i}$. Then the above expression is bounded by

$$
\begin{aligned}
&|\hat{\theta}(m_2)\cdots\hat{\theta}(m_{M-1})|^2 \sum_{m_0\in\mathbb{Z}^J} \prod_{i=1}^{J} |m_{0,i} + m_{2,i} + \cdots + m_{p_i,i}|\,|\hat{\theta}(m_0)|^2 \\
&\quad\le |\hat{\theta}(m_2)\cdots\hat{\theta}(m_{M-1})|^2 \\
&\qquad\times \sum_{m_0\in\mathbb{Z}^J} (|m_0| + |m_2| + \cdots + |m_{M-1}|)^J\,|\hat{\theta}(m_0)|^2 \\
&\quad\le |\hat{\theta}(m_2)\cdots\hat{\theta}(m_{M-1})|^2 \\
&\qquad\times \sum_{m_0\in\mathbb{Z}^J} \left((M-1)^J|m_0|^J + \sum_{i=2}^{M-1}(M-1)^J|m_i|^J\right)|\hat{\theta}(m_0)|^2 \\
&\quad\le |\hat{\theta}(m_2)\cdots\hat{\theta}(m_{M-1})|^2 (M-1)^J c_1\left(1 + \sum_{i=2}^{M-1}|m_i|^J\right). \qquad \blacksquare
\end{aligned}
$$

Proof of (2). In general if $A = BC$ where B and C are Hilbert–Schmidt operators then $\|A\|_1 \le \|B\|_2\,\|C\|_2$. Therefore

$$
\begin{aligned}
&\sum_{m_2,\ldots,m_{M-1}\in\mathbb{Z}^J} \|A_p(m_2,\ldots,m_{M-1})\|_1 \\
&\quad\le \sum_{m_2,\ldots,m_{M-1}\in\mathbb{Z}^J} \left(\sum_{m,k\in\mathbb{Z}_+^J} |P_{m_2,\ldots,m_{M-1}}(m,k)|^2\right)^{1/2} \\
&\qquad\times\left(\sum_{m,k\in\mathbb{Z}_+^J} |Q_{m_2,\ldots,m_{M-1}}(m,k)|^2\right)^{1/2} \\
&\quad\le \sum_{m_2,\ldots,m_{M-1}\in\mathbb{Z}^J} c_1(M-1)^J\left(1 + \sum_{i=2}^{M-1}|m_i|^J\right)|\hat{\theta}(m_2)\cdots\hat{\theta}(m_{M-1})|
\end{aligned}
$$

(by claims (i) and (ii))

$$
\begin{aligned}
&= c_1(M-1)^J\left(\left(\sum_{m\in\mathbb{Z}^J}|\hat{\theta}(m)|\right)^{M-2} + (M-2)\right. \\
&\qquad\qquad \left.\times\left(\sum_{m\in\mathbb{Z}^J}|\hat{\theta}(m)|\right)^{M-3}\sum_{m\in\mathbb{Z}^J}|\hat{\theta}(m)|\,|m|^J\right)
\end{aligned}
$$

$$
\begin{aligned}
&= c_1(M-1)^J\Bigl(\sum_{m\in\mathbb{Z}^J}|\hat\theta(m)|\Bigr)^{M-3}\\
&\qquad\times\Bigl(\sum_{m\in\mathbb{Z}^J}|\hat\theta(m)| + (M-2)\sum_{m\in\mathbb{Z}^J}|\hat\theta(m)|\,|m|^J\Bigr)\\
&\le c_1(M-1)^J\Bigl(\sum_{m\in\mathbb{Z}^J}|\hat\theta(m)|\Bigr)^{M-3}\\
&\qquad\times\Bigl((M-1)\sum_{m\in\mathbb{Z}^J}|\hat\theta(m)| + (M-1)\sum_{m\in\mathbb{Z}^J}|\hat\theta(m)|\,|m|^J\Bigr)\\
&\le c_1^2(M-1)^{J+1}\Bigl(\sum_{m\in\mathbb{Z}^J}|\hat\theta(m)|\Bigr)^{M-3}. \qquad\blacksquare
\end{aligned}
$$

We have just shown that each operator $A_{\phi_{J,d,t},F}$ is of trace class. The next step is to find a formula for $\int_{P_J^d}\operatorname{tr}(A_{\phi_{J,d,t},F})\,dt$. The result is

LEMMA 3.7. *Let $F(z)=z^M$ and let ϕ: $L^2(\mathbb{T}^J)\to\mathbb{C}$ satisfy condition $(*)$, then*

$$
\begin{aligned}
\Bigl(\frac{1}{2\pi}\Bigr)^{N-J}\int_{P_J^d}\operatorname{tr}(A_{\phi_{J,d,t},F})\,dt &= \sum_{m\in\mathbb{Z}^N}\hat\phi(m_1)\cdots\hat\phi(m_{M-1})\\
&\qquad\times\hat\phi(-m_1-\cdots-m_{M-1})\\
&\qquad\times\prod_{i=1}^{J}\max\bigl\{0, m_1\cdot\nu_{d_i}, (m_1+m_2)\cdot\nu_{d_i},\dots,(m_1+\cdots+m_{M-1})\cdot\nu_{d_i}\bigr\}.
\end{aligned}
$$

Proof. Since $A_{\phi_{J,d,t},F}$ is of trace class and since its kernel (found in Lemma 3.4) is continuous we can find its trace by summing the kernel over the diagonal of $\mathbb{Z}_+^J\times\mathbb{Z}_+^J$. We obtain the formula

$$
\begin{aligned}
\int_{p_J^d}\operatorname{tr}(A_{\phi_{J,d,t},F})\,dt &= \int_{p_J^d}\Bigl(\sum_{m\in\mathbb{Z}_+^J}\ \sum_{m_1,\dots,m_{M-1}\in\mathbb{Z}^J}\hat\phi_{J,d,t}(m_1)\cdots\hat\phi_{J,d,t}(m_{M-1})\\
&\qquad\times\hat\phi_{J,d,t}(-m_1-\cdots-m_{M-1})\\
&\qquad\times\prod_{i=1}^{J}\bigl(1-\chi_i(m-m_1)\cdots\chi_i(m-m_1-\cdots-m_{M-1})\bigr)\Bigr)\,dt.
\end{aligned}
$$

Since ϕ satisfies condition $(*)$, Fubini's theorem applies and the above expression is equal to

$$\sum_{m_1,\dots,m_{M-1}\in\mathbb{Z}^J}\int_{P_J^d}\Big(\hat{\phi}_{J,d,t}(m_1)\cdots\hat{\phi}_{J,d,t}(m_{M-1})\times\hat{\phi}_{J,d,t}(-m_1-\cdots-m_{M-1})\Big)\,dt$$
$$\times\prod_{i=1}^{J}\max\{0,m_{1,i},(m_{1,i}+m_{2,i}),\dots,(m_{1,i}+\cdots+m_{M-1,i})\}.$$

Make the following substitutions:

$$\begin{aligned}g_1(t)&=\hat{\phi}_{J,d,t}(m_1)\\ g_2(t)&=\hat{\phi}_{J,d,t}(m_2)\\ &\cdots\\ g_{M-1}(t)&=\hat{\phi}_{J,d,t}(m_{M-1})\\ g_M(t)&=\hat{\phi}_{J,d,t}(-m_1-\cdots-m_{M-1}).\end{aligned}$$

Then by identifying P_J^d with $\mathbb{T}^{N-J}$ our expression is equal to

$$\sum_{m_1,\dots,m_{M-1}\in\mathbb{Z}^J}\int_{\mathbb{T}^{N-J}}g_1(t)g_2(t)\cdots g_M(t)\,dt$$
$$\times\prod_{i=1}^{J}\max\{0,m_{1,i},(m_{1,i}+m_{2,i}),\dots,(m_{1,i}+\cdots+m_{M-1,i})\}$$
$$=(2\pi)^{N-J}\sum_{m_1,\dots,m_{M-1}\in\mathbb{Z}^J}(g_1g_2\cdots g_M)^{\wedge}(0)$$
$$\times\prod_{i=1}^{J}\max\{0,m_{1,i},(m_{1,i}+m_{2,i}),\dots,(m_{1,i}+\cdots+m_{M-1,i})\}$$
$$=(2\pi)^{N-J}\sum_{m_1,\dots,m_{M-1}\in\mathbb{Z}^J}\hat{g}_1*\hat{g}_2*\cdots*\hat{g}_M(0)$$
$$\times\prod_{i=1}^{J}\max\{0,m_{1,i},(m_{1,i}+m_{2,i}),\dots,(m_{1,i}+\cdots+m_{M-1,i})\}$$
$$=(2\pi)^{N-J}\sum_{m_1,\dots,m_{M-1}\in\mathbb{Z}^J}\ \sum_{n_1,\dots,n_{M-1}\in\mathbb{Z}^{N-J}}\hat{g}_1(n_1)\hat{g}_2(n_2)\cdots$$
$$\cdots\hat{g}_{M-1}(n_{M-1})\hat{g}_M(-n_1-\cdots-n_{M-1})$$
$$\times\prod_{i=1}^{J}\max\{0,m_{1,i}(m_{1,i}+m_{2,i}),\dots,(m_{1,i}+\cdots+m_{M-1,i})\}.$$

By identifying $n \in \mathbb{Z}^{N-J}$ with $n \in \{\nu_{d_1}, \dots, \nu_{d_J}\}^{\perp}$ it can be shown that $\hat{g}_k(n_k) = \hat{\phi}(n_k + \sum_{i=1}^{J} m_{k,i}\nu_{d_i})$ for $k = 1, 2, \dots, M-1$ and $\hat{g}_M(n) = \hat{\phi}(n + \sum_{i=1}^{J}(-l_{1,i} - l_{2,i} - \cdots - l_{M-1,i})\nu_{d_i})$. By letting $l_k = n_k + \sum_{i=1}^{J} m_{k,i}\nu_{d_i}$, we get that $m_{k,i} = l_k \cdot \nu_{d_i}$ and our expression is equal to

$$(2\pi)^{N-J} \sum_{l_1, \dots, l_{M-1} \in \mathbb{Z}^N} \hat{\phi}(l_1) \cdots \hat{\phi}(l_{M-1}) \hat{\phi}(-l_1 - \cdots - l_{M-1})$$
$$\times \prod_{i=1}^{J} \max\left\{0, l_1 \cdot \nu_{d_i}, (l_1 + l_2) \cdot \nu_{d_i}, \dots, (l_1 + \cdots + l_{M-1}) \cdot \nu_{d_i}\right\}. \quad \blacksquare$$

By applying Lemma 3.7 we can now find a formula for the sum $\sum_{J=0}^{N} c_{J,F} p^{N-J}$.

LEMMA 3.8. *Let $F(z) = z^M$ and let ϕ: $\mathbb{T}^N \to \mathbb{C}$ satisfy condition* $(*)$. *Then*

$$\sum_{J=0}^{N} c_{J,F} p^{N-J} = \sum_{m_1, \dots, m_{M-1} \in \mathbb{Z}^N} \hat{\phi}(m_1) \cdots \hat{\phi}(m_{M-1})$$
$$\times \hat{\phi}(-m_1 - \cdots - m_{M-1}) \prod_{k=1}^{N} (pa_k - Z_k)$$

where

$$c_{J,F} = (-1)^J \left(\frac{1}{2\pi}\right)^{N-J} \sum_{\Lambda_J^d \in \Lambda_J} |\Lambda_J^d| \int_{p_j^d} \operatorname{tr}\left(A_{\phi_{J,d,t},F}\right) dt.$$

the a_k are the number of lattice points along the kth edge of Λ, and if μ_k is the standard unit vector in $\mathbb{C}^N$ in the direction of the edge a_k then

$$Z_k = \max\left\{0, m_1 \cdot \mu_k, (m_1 + m_2) \cdot \mu_k, \dots, (m_1 + \cdots + m_{M-1}) \cdot \mu_k\right\}$$
$$+ \max\left\{0, -m_1 \cdot \mu_k, (-m_1 - m_2) \cdot \mu_k, \right.$$
$$\left. \dots, (-m_1 - \cdots - m_{M-1}) \cdot \mu_k\right\}.$$

Proof.

$$\prod_{k=1}^{N} (pa_k - Z_k) = p^N(a_1 a_2 \cdots a_N) + \sum_{J=1}^{N} (-1)^J p^{N-J}$$
$$\times \sum_{1 \le k_1 < \cdots < k_J \le N} \left(\prod_{k \ne k_1, \dots, k_J} a_k\right) Z_{k_1} Z_{k_2} \cdots Z_{k_J}$$

$$= p^N |\Lambda| + \sum_{J=1}^{N} (-1)^J p^{N-J} \sum_{\Lambda_J^d \in \Lambda_J} |\Lambda_J^d|$$
$$\times \prod_{i=1}^{J} \max\{0, m_1 \cdot \nu_{d_i}, (m_1 + m_2) \cdot \nu_{d_i},$$
$$\ldots, (m_1 + \cdots + m_{M-1}) \cdot \nu_{d_i}\}.$$

Therefore

$$\sum_{J=0}^{N} c_{J,F} = \sum_{J=0}^{N} p^{N-J}(-1)^J \sum_{\Lambda_J^d \in \Lambda_J} |\Lambda_J^d| \left(\left(\frac{1}{2\pi} \right)^{N-J} \int_{p_j^d} \operatorname{tr}\left(A_{\phi_{J,d,t},F} \right) dt \right)$$
$$= \sum_{J=0}^{N} (-1)^J p^{N-J} \sum_{\Lambda_J^d \in \Lambda_J} |\Lambda_J^d|$$
$$\times \left(\sum_{m_1, \ldots, m_{M-1} \in \mathbb{Z}^N} \hat{\phi}(m_1) \cdots \hat{\phi}(m_{M-1}) \right.$$
$$\times \hat{\phi}(-m_1 - \cdots - m_{M-1})$$
$$\left. \times \prod_{i=1}^{J} \max\{0, m_1 \cdot \nu_{d_i}, (m_1 + m_2) \cdot \nu_{d_i}, \ldots, (m_1 + \cdots + m_{M-1}) \cdot \nu_{d_i}\} \right)$$
$$= \sum_{m_1, \ldots, m_{M-1} \in \mathbb{Z}^J} \hat{\phi}(m_1) \cdots \hat{\phi}(m_{M-1})$$
$$\times \hat{\phi}(-m_1 - \cdots - m_{M-1}) \prod_{k=1}^{N} (pa_k - Z_k). \qquad \blacksquare$$

At this point by making use of Lemmas 3.1 and 3.8 we are able to prove Theorem 1 for the case when $F(z) = z^M$.

LEMMA 3.9. *Let $F(z) = z^M$ and let $\phi \colon \mathbb{T}^N \to \mathbb{C}$ satisfy $(*)$. Then*

$$\left| \operatorname{tr}\left(T_p(\phi) \right)^M - \sum_{J=0}^{N} c_{J,F} p^{N-J} \right| \le c(M-1)^{N+1} \|\hat{\phi}\|_1^M$$
$$\times \sum_{|m| \ge pa/(M-1)} |\hat{\phi}(m)| \, |m|^N$$

where a and c are constants independent of p and M.

Proof. Let $\tilde{\Lambda} = \Lambda_p \cap (\Lambda_p + m_1) \cap \cdots \cap (\Lambda_p + m_1 + \cdots + m_{M-1})$. Then by Lemma 3.1

$$\operatorname{tr}\big(T_p(\phi)\big)^M = \sum_{m_1, \dots, m_{M-1} \in \mathbb{Z}^J} \hat{\phi}(m_1) \cdots \hat{\phi}(m_{M-1}) \times \hat{\phi}(-m_1 - \cdots - m_{M-1}) |\tilde{\Lambda}|$$

and by Lemma 3.8

$$\sum_{J=0}^{N} c_{J,F} p^{N-J} = \sum_{m_1, \dots, m_{M-1} \in \mathbb{Z}^N} \hat{\phi}(m_1) \cdots \hat{\phi}(m_{M-1}) \times \hat{\phi}(-m - \cdots - m_{M-1}) \prod_{k=1}^{N} (pa_k - Z_k).$$

Therefore

$$\left| \operatorname{tr}\big(T_p(\phi)\big)^M - \sum_{J=0}^{N} c_{J,F} p^{N-J} \right| \le \sum_{m_1, \dots, m_{M-1} \in \mathbb{Z}^N} \times \left| \hat{\phi}(m_1) \cdots \hat{\phi}(m_{M-1}) \hat{\phi}(-m_1 - \cdots - m_{M-1}) \right| \times \left| |\tilde{\Lambda}| - \prod_{k=1}^{N} (pa_k - Z_k) \right|. \quad (3.5)$$

Before continuing we need to note some properties about the cardinality of $\tilde{\Lambda}$.

(i) $|\tilde{\Lambda}| = \prod_{k=1}^{N} (pa_k - Z_k)$ whenever $\tilde{\Lambda} \neq \emptyset$,

(ii) if $|m_i| < pa/(M-1)$ for all $i \in \{1, 2, \dots, M-1\}$ where $a = \min\{a_1, \dots, a_N\}$ then $\tilde{\Lambda} \neq \emptyset$.

Let $E = \{(m_1, \dots, m_{M-1}) \in (\mathbb{Z}^N)^{M-1} \colon \max |m_i| \ge pa/(M-1)\}$. Then by (ii) we know $\tilde{\Lambda} \neq \emptyset$ on the complement of E. Therefore by (i) expression (3.5) is equal to

$$\sum_{E} \left| \hat{\phi}(m_1) \cdots \hat{\phi}(m_{M-1}) \hat{\phi}(-m_1 - \cdots - m_{M-1}) \prod_{k=1}^{N} (pa_k - Z_k) \right|.$$

Define $E_j = \{(m_1, \dots, m_{M-1}) \in (\mathbb{Z}^N)^{M-1} \colon \ |m_j| = \max |m_i| \ge pa/(M-1)\}$. Then $E = \bigcup_{j=1}^{M-1} E_j$ and our expression is bounded by

$$\sum_{j=1}^{M-1} \sum_{E_J} \left| \hat{\phi}(m_1) \cdots \hat{\phi}(m_{M-1}) \hat{\phi}(-m_1 - \cdots - m_{M-1}) \prod_{k=1}^{N} (pa_k - Z_k) \right|.$$

Next by letting $A = \max\{a_k\}$ and by noting that $|Z_k| \leq 2(|m_1| + \cdots + |m_{M-1}|)$ it follows that this is bounded by

$$\sum_{j=1}^{M-1} \sum_{E_j} |\hat{\phi}(m_1) \cdots \hat{\phi}(m_{M-1}) \hat{\phi}(-m_1 - \cdots m_{M-1})|$$

$$\times \big(pA + 2(|m_1| + \cdots + |m_{M-1}|)\big)^N$$

$$\leq \sum_{j=1}^{M-1} \sum_{E_J} |\hat{\phi}(m_1) \cdots \hat{\phi}(m_{M-1}) \hat{\phi}(-m_1 - \cdots - m_{M-1})|$$

$$\times \left(\frac{A}{a}(M-1)|m_j| + 2(M-1)|m_j| \right)^N$$

(since $|m_i| \geq pa/(M-1) \Rightarrow pA \leq (A/a)(M-1)|m_i|$ and since $|m_j| = \max\{|m_i|\}$)

$$= \sum_{j=1}^{M-1} \sum_{E_j} |\phi(m_1) \cdots \hat{\phi}(m_{M-1}) \hat{\phi}(-m_1 - \cdots - m_{M-1})|$$

$$\times \left((M-1)|m_j| \left(\frac{A}{a} + 2 \right) \right)^N$$

$$= \left(\frac{A}{a} + 2 \right)^N (M-1)^{N+1} \sum_{E_1} |\hat{\phi}(m_1) \cdots \hat{\phi}(m_{M-1})$$

$$\times \hat{\phi}(-m_1 - \cdots - m_{M-1})| \, |m_1|^N$$

$$\leq \left(\frac{A}{a} + 2 \right)^N (M-1)^{N+1} \sum_{m_1, \ldots, m_{M-1} \in \mathbb{Z}^N} \sum_{|m_1| \geq pa/(M-1)} |\hat{\phi}(m_1) \cdots$$

$$\cdots \hat{\phi}(m_{M-1}) \hat{\phi}(-m_1 - \cdots - m_{M-1})| \, |m_1|^N$$

$$\leq \left(\frac{A}{a} + 2 \right)^N (M-1)^{N+1} \|\hat{\phi}\|_1^{M-2} \|\hat{\phi}\|_\infty \sum_{|m_1| \geq pa/(M-1)} |\hat{\phi}(m_1)| \, |m_1|^N$$

$$= c(M-1)^{N+1} \|\hat{\phi}\|_1^M \sum_{|m| \geq pa/(M-1)} |\hat{\phi}(m)| \, |m|^N. \qquad \blacksquare$$

Finally, by applying Lemma 3.9 we can prove Theorem 1.

Proof of Theorem 1. In order to prove the theorem we need to show that

$$\lim_{p\to\infty}\left|\operatorname{tr} F\big(T_p(\phi)\big) - \sum_{J=0}^{N} c_{J,F}\, p^{N-J}\right| = 0.$$

Express F as a power series, $F(z) = \sum_{m=0}^{\infty} a_m z^m$, which converges absolutely and uniformly for $|z| < \|\hat{\phi}\|_1 + \delta$. Let $F_M(z) = \sum_{m=0}^{M} a_m z^m$. Then

$$\begin{aligned}
&\left|\operatorname{tr} F\big(T_p(\phi)\big) - \sum_{J=0}^{N} c_{J,F}\, p^{N-J}\right| \\
&\quad = \lim_{M\to\infty}\left|\operatorname{tr} F_M\big(T_p(\phi)\big) - \sum_{J=0}^{N} c_{J,F_M}\, p^{N-J}\right| \\
&\quad = \lim_{M\to\infty}\left|\sum_{m=0}^{M} a_m\left(\operatorname{tr}\big(T_p(\phi)\big)^m - \sum_{J=0}^{N} c_{J,z^m}\, p^{N-J}\right)\right| \\
&\quad \le \sum_{m=0}^{\infty} |a_m|\left|\operatorname{tr}\big(T_p(\phi)\big)^m - \sum_{J=0}^{N} c_{J,z^m}\, p^{N-J}\right| \\
&\quad = \sum_{m=2}^{\infty} |a_m|\left|\operatorname{tr}\big(T_p(\phi)\big)^m - \sum_{J=0}^{N} c_{J,z^m}\, p^{N-J}\right| \\
&\quad \le \sum_{m=1}^{\infty} |a_m|\, c(M-1)^{N+1}\,\|\hat{\phi}\|_1^M \sum_{|m|\ge pa/(M-1)} |\phi\,\hat{}\,(m)|\,|m|^N \\
&\qquad\quad (\text{by Lemma 3.9}). \qquad (3.6)
\end{aligned}$$

By the Lebesgue Dominated Convergence Theorem (LDCT) $\lim_{p\to\infty}$ (3.6) $=$

$$\begin{aligned}
&\sum_{m=1}^{\infty} |a_m|\, c(M-1)^{N+1}\,\|\hat{\phi}\|_1^M\left(\lim_{p\to\infty}\sum_{|m|\ge pa/(M-1)} |\hat{\phi}(m)|\,|m|^N\right) \\
&\quad = \sum_{m=1}^{\infty} |a_m|\, c(M-1)^{N+1}\,\|\hat{\phi}\|_1^M\left(\lim_{p\to\infty}\sum_{\mathbb{Z}^N} |\hat{\phi}(m)|\,|m|^N \chi_{E_p}(m)\right)
\end{aligned}$$

where

$$\begin{aligned}
E_p &= \left\{m : |m| \ge \frac{pa}{M-1}\right\} \\
&= \sum_{m=1}^{\infty} |a_m|\, c(M-1)^{N+1}\,\|\hat{\phi}\|_1^M\left(\sum_{\mathbb{Z}^N}\lim_{p\to\infty} |\hat{\phi}(m)|\,|m|^N \chi_{E_p}(m)\right)
\end{aligned}$$

(by the LDCT) $= 0$. ∎

3. A GENERALIZATION OF THEOREM 1 AND AN EXAMPLE

In Theorem 1 we assumed that the edges of the rectangular lattice Λ_p all expanded at the same rate. With minor changes to the proof we can drop this assumption. Let $p = (p_1, \ldots, p_N) \in (\mathbb{Z}_+ \setminus 0)^N$ then

$$\Lambda_p = \left\{(m_1, \ldots, m_N) \in \mathbb{Z}_+^N : m_i \in \left[0, p_i a_i\right)\right\}$$

and we get the following generalization of Theorem 1.

THEOREM 2. *Suppose* $\phi\colon \mathbb{T}^N \to \mathbb{C}$ *satisfies condition* $(*)$ *and suppose* F *is analytic on* $D(0, \|\hat{\phi}\|_1 + \delta)$ *for some* $\delta > 0$. *Then*

$$\operatorname{tr}\Big(F\big(T_p(\phi)\big)\Big) = \sum_{J=0}^{N} c_{J,F}(p) + o(1) \qquad \text{as } \inf p_i \to \infty$$

where

$$c_{J,F}(p) = (-1)^J \left(\frac{1}{2\pi}\right)^{N-J} \sum_{(\Lambda_p)_J^d \in (\Lambda_p)_J} \left|(\Lambda_p)_J^d\right| \int_{P_J^d} \operatorname{tr}\left(A_{\phi_{J,d,t},F}\right) dt.$$

Theorem 2 enables us to explicitly compute the first $N+1$ terms of the asymptotic expansion for the trace of $(T_p(\phi))^{-1}$ as $\inf p_i \to \infty$ where ϕ satisfies certain properties. By carefully applying the theorem to the function $\sigma = \phi - \hat{\phi}(0)$ we obtain the following result:

THEOREM 3. *Let* $\phi = gh$ *where* $g^{\pm 1}$ *and* $(\bar{h})^{\pm 1} \in H^\infty(\mathbb{T}^N)$. *Assume also that* ϕ *satisfies condition* $(*)$ *and that*

$$|\hat{\phi}(0)| > \sum_{m \in \mathbb{Z}^N \setminus 0} |\hat{\phi}(m)|.$$

Then

$$\operatorname{tr}\big(T_p(\phi)\big)^{-1} = \sum_{m \in \mathbb{Z}_+^N} \left(\frac{1}{g}\right)^{\wedge}(m) \left(\frac{1}{h}\right)^{\wedge}(-m) \prod_{i=1}^{N} (a_i p_i - m_i) + o(1).$$

Proof. Let $\sigma = \phi - \hat{\phi}(0)$ and let $F(z) = (z + \hat{\phi}(0))^{-1}$. Then σ satisfies $(*)$ and F is analytic on $D(0, \|\hat{\sigma}\|_1 + \delta)$ for some $\delta > 0$ since

$$\|\hat{\sigma}\|_1 = \sum_{m \in \mathbb{Z}^N \setminus 0} |\hat{\phi}(m)| < |\hat{\phi}(0)|.$$

Therefore by Theorem 2

$$\operatorname{tr}\Big(F\big(T_p(\phi)\big)\Big) = \sum_{J=0}^{N} c_{J,F}(p) + o(1) \qquad \text{as } \inf p_i \to \infty$$

where

$$c_{J,F}(p) = (-1)^J \left(\frac{1}{2\pi}\right)^{N-J} \sum_{(\Lambda_p)_J^d \in (\Lambda_p)_J} |(\Lambda_p)_J^d| \int_{p_J^d} \operatorname{tr}\big(A_{\phi_{J,d,t},F}\big)\, dt.$$

So to prove Theorem 3 it suffices to show that

$$\sum_{J=0}^{N} c_{J,F}(p) = \sum_{m \in \mathbb{Z}_+^N} \left(\frac{1}{g}\right)^{\wedge}(m) \left(\frac{1}{h}\right)^{\wedge}(-m) \prod_{i=1}^{N} (a_i p_i - m_i).$$

The fact that $g^{\pm 1}$ and $(\bar{h})^{\pm 1} \in H^\infty(\mathbb{T}^N)$ implies that

$$F(T(\sigma)) = \big(T(\sigma) + \hat{\phi}(0)\big)^{-1} = (T(gh))^{-1} = (T(h)T(g))^{-1}$$

$$= (T(g))^{-1}(T(h))^{-1} = T\left(\frac{1}{g}\right)T\left(\frac{1}{h}\right).$$

In Lemma 3.8 we showed that if $F(T(\sigma)) = T(\sigma)T(\sigma)$ then

$$\sum_{J=0}^{N} c_{J,F}(p) = \sum_{m \in \mathbb{Z}^N} \hat{\phi}(m)\, \hat{\sigma}(-m) \prod_{i=1}^{N} (a_i p_i - Z_i)$$

where $Z_i = \max\{0, m \cdot \mu_i\} + \max\{0, -m \cdot \mu_i\} = |m_i|$.

Here we replace $F(T(\sigma)) = T(\sigma)T(\sigma)$ with $F(T(\sigma)) = T(1/g)T(1/h)$. By the same argument we get

$$\sum_{J=0}^{N} c_{J,F}(p) = \sum_{m \in \mathbb{Z}^N} \left(\frac{1}{g}\right)^{\wedge}(m) \left(\frac{1}{h}\right)^{\wedge}(-m) \prod_{i=1}^{N} \big(a_i p_i - |m_i|\big).$$

The result follows from the fact that $1/g \in H^\infty(\mathbb{T}^N)$. ∎

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